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ASYMPTOTIC EFFICIENCY OF THE KOLMOGOROV - SMIRNOV TEST

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A simple derivation of asymptotic efficiency for the Kolmogorov - Smirnov statistic is given and evaluated for normal location and normal scale alternatives. Using equal samples to simplify the derivation, the limiting efficiency is obtained by letting the type I error α go to zero while the type II error goes to β, $0 < \beta < 1$. For symmetric location alternatives, the efficiency is the same as that obtained for the Mood and Brown median test. Limits of relative efficiencies for alternatives which approach the null hypothesis are $2/\pi$ for normal location alternatives and $(\pi e)^{-1}$ for normal scale alternatives.

I. INTRODUCTION

Let $X_1, X_2, \ldots, X_m$ be independent with cumulative distribution function $F(x)$ and let $Y_1, Y_2, \ldots, Y_n$ be independent with c.d.f. $G(x)$. To test the hypothesis of equality of F and G, the Kolmogorov - Smirnov statistics

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$$D = \sup_x | F_m(x) - G_n(x) |, \quad \text{and} \quad D^+ = \sup_x (F_m(x) - G_n(x))$$

where F_m and G_n are the sample c.d.f.s are often recommended.

In a recent article by Capon [5], bounds for limiting Pitman efficiency were derived. The purpose of this paper is to extend the asymptotic comparisons by employing a different limiting efficiency as defined by Bahadur [3, p. 87].

With Pitman efficiency, the limiting ratio of sample sizes is derived with sample sizes, critical values, and alternatives adjusted so that both tests obtain limiting type I and type II errors α, β with $0 < \alpha, \beta < 1$. For the exact Bahadur efficiency which we consider, the alternative is kept fixed and critical values are adjusted so that the type II error approaches β with $0 < \beta < 1$ and the type I error goes to zero (at an exponential rate) with increasing sample size. The exact Bahadur efficiency appears generally more informative than the Pitman efficiency as it depends upon the alternative. For those cases where both efficiencies have been computed, the exact Bahadur efficiency yields the Pitman value as a limit when the alternative approaches the null hypothesis. For example, see Bahadur [3] and Klotz [8].

II. KOLMOGOROV - SMIRNOV COMPUTATIONS

For simplicity we restrict attention to the case of equal samples $m = n$ and the statistic D^+. For alternatives F, G, we reject the hypothesis $F = G$ if $D^+ > \rho_n$. We first show that the critical value

ρ_n converges to ρ where $\rho = \sup_x (F(x) - G(x))$ in order to have

$$\beta_n = P_{F,G}[D^+ \leqq \rho_n] \to \beta, \quad \text{with } 0 < \beta < 1, \text{ as } n \to \infty.$$

We show by contradiction that $\rho \leqq \liminf \rho_n \leqq \limsup \rho_n \leqq \rho$. Assume first that $\limsup \rho_n > \rho$. Under this assumption there exists a subsequence $\{n'\}$ for which $\rho_{n'} \to \limsup \rho_n > c > \rho$ and

$$\beta_{n'} = P[D^+ \leqq \rho_{n'}] \geqq P[\rho + U_{n'} + V_{n'} \leqq \rho_{n'}]$$

$$= P[U_{n'} + V_{n'} \leqq \rho_{n'} - \rho] \geqq P[U_{n'} + V_{n'} \leqq c - \rho]$$

which follows by writing

$$F_n - G_n = F - G + F_n - F + G - G_n$$

so that

$$\sup_x (F_n(x) - G_n(x)) \leqq \rho + U_n + V_n.$$

Here $U_n = \sup_x (F_n(x) - F(x))$, $V_n = \sup (G(x) - G_n(x))$, and ρ is given above. Thus we have the contradiction that

$$\lim_{n'} \beta_{n'} \geqq \lim_{n'} P[U_{n'} + V_{n'} \leqq c - \rho] = 1$$

Since $c - \rho > 0$ and $U_{n'} \xrightarrow{P} 0$, $V_{n'} \xrightarrow{P} 0$ by the Glevenko - Cantelli Lemma [9, p. 20]. We next show $\rho \leqq \liminf \rho_n$. Assume the converse $\liminf \rho_n < \rho$ and write

$$F - G = F_n - G_n + F - F_n + G_n - G$$

so that

$$\rho \leqq D^+_{nn} + W_n + Z_n$$

with

$$W_n = \sup_x (F(x) - F_n(x)), \quad Z_n = \sup_x (G_n(x) - G(x)).$$

For a subsequence $\{n''\}$ for which $\rho_{n''} \to \liminf \rho_n < c < \rho$ (which exists by our assumption) we have

$$\beta_{n''} = P[\, D^+ \leqq \rho_{n''}\,] \leqq P[\, \rho - (W_{n''} + Z_{n''}) \leqq \rho_{n''}\,]$$

$$= P[\, W_{n''} + Z_{n''} \geqq \rho - \rho_{n''}\,] \leqq P[\, W_{n''} + Z_{n''} \geqq \rho - c\,].$$

The contradiction follows from

$$\lim_{n''} \beta_{n''} \leqq \lim_{n''} P[\, W_{n''} + Z_{n''} \geqq \rho - c > 0\,] = 0$$

which is a consequence of $W_{n''}, Z_{n''} \xrightarrow{P} 0$ using the Glevenko - Cantelli Lemma again.

Next it is known (see for example Hodges [7]) that the principle of reflection gives the null distribution for equal samples

$$\alpha_n = P[\, D^+ > \rho_n\,] = \binom{2n}{n\rho_n} \Big/ \binom{2n}{n}.$$

Using Stirling's approximation in the combinatorials and considering alternatives F, G for which $0 < \rho < 1$ we obtain using $\lim \rho_n = \rho$

$$\lim_{n\to\infty} -\frac{1}{n} \ln \alpha_n = (1-\rho)\ln(1-\rho) + (1+\rho)\ln(1+\rho). \tag{2.1}$$

If F and G have symmetric densities with $G(x) = F(x-\Delta)$, then $\rho = 2F(\Delta/2) - 1$. For normal location alternatives $F(x) = \Phi(x)$, $G(x) = \Phi(x-\Delta)$ the expression (2.1) reduces to

$$2\,\Phi(\Delta/2)\ln 2\,\Phi(\Delta/2) + 2\,\Phi(-\Delta/2)\ln 2\,\Phi(-\Delta/2) \tag{2.2}$$

Similarly for normal scale alternatives $F(x) = \Phi(x/\sigma)$, $G(x) = \Phi(x/\tau)$ if we denote $\theta = \tau/\sigma$ with $\theta > 1$ we have

$$\rho = \Phi\left(\theta\sqrt{\frac{2\ln\theta}{\theta^2-1}}\right) - \Phi\left(\sqrt{\frac{2\ln\theta}{\theta^2-1}}\right) \tag{2.3}$$

III. PARAMETRIC COMPUTATIONS

For the case of normal shift alternatives, the appropriate parametric test for comparison with the Kolmogorov - Smirnov test is the two sample t test. With equal samples, we reject if

$$t = \sqrt{\frac{n}{2}}\,(\bar{y} - \bar{x})/S > C_n$$

where

$$S^2 = \left[\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{n}(y_j - \bar{y})^2\right] / (2n-2)$$

In order for the type II error β_n to converge to β with $0 < \beta < 1$ it it sufficient that the critical value satisfy $C_n = \sqrt{\frac{n}{2}}\,\Delta + W$ for alternatives $F(x) = \Phi(x - \Delta)$. This is shown by writing

$$\beta_n = P[\sqrt{\tfrac{n}{2}}\,(y - \bar{x})/S < C_n] = P[\sqrt{\tfrac{n}{2}}\,(\bar{y} - \bar{x}) < (\sqrt{\tfrac{n}{2}}\,\Delta + W)S]$$

$$= P[\sqrt{\tfrac{n}{2}}\,(\bar{y} - \bar{x} - \Delta) < \sqrt{\tfrac{n}{2}}\,\Delta(S-1) + WS]$$

$$= \int_0^{\infty} \Phi(\overline{\tfrac{n}{2}}\,\Delta(s-1) + Ws)\, d\,F_S(s). \tag{3.1}$$

If we denote the random variable $U_n = 2\sqrt{n}\,(S-1)$ then we know U_n has a limiting normal distribution. Changing variables and using the Helly Bray theorem [9, p. 182] the expression (3.1) becomes

$$\beta_n = \int_{-\infty}^{\infty} \Phi(\frac{u\Delta}{\sqrt{8}} + W(1 + \frac{u}{2\sqrt{n}}))\, d\,F_{U_n}(u) \rightarrow \beta$$

where

$$\beta = \int_{-\infty}^{\infty} \Phi(\frac{u\Delta}{\sqrt{8}} + W)\, d\,\Phi(u) \quad \text{and} \quad 0 < \beta < 1.$$

We next show

$$\lim_{n\to\infty} -\frac{1}{n} \log \alpha_n = \log(1 + (\tfrac{\Delta}{2})^2). \tag{3.2}$$

With the critical value $C_n = \sqrt{\frac{n}{2}}\,\Delta + W$ we have under the null hypothesis

$$\alpha_n = P[\,t_{2n-2} > C_n\,]$$

$$= \frac{\Gamma((2n-1)/2)}{\sqrt{(2n-2)\pi}\;\Gamma((2n-3)/2)} \int_{C_n}^{\infty} \frac{dt}{(1+\frac{t^2}{2n-2})^{(2n-1)/2}}$$

$$= \frac{2n-2}{2n-3}\left(1+\frac{C_n^2}{2n-2}\right)^{-(2n-3)/2} \frac{1}{C_n} \qquad (3.3)$$

$$- \frac{(2n-2)}{(2n-3)(2n-5)}\left(1+\frac{C_n^2}{2n-2}\right)^{-\left(\frac{2n-3}{2}\right)} \frac{1}{C_n}\left(1+\frac{2n-2}{C_n^2}\right) + R_2,$$

where

$$|R_2| \leq \frac{(2n-2)}{(2n-3)(2n-5)} \frac{1}{C_n}\left(1+\frac{2n-2}{C_n^2}\right)\left(1+\frac{C_n^2}{2n-2}\right)^{-\left(\frac{2n-3}{2}\right)}.$$

The expression (3.3) is obtained by using two terms in the mills ratio expansion for the t-distribution derived by Pinkham and Wilk [11]. Thus the expression (3.2) follows by substituting $\sqrt{\frac{n}{2}}\,\Delta + W$ for C_n in (3.3) and taking the limit.

For normal scale alternatives the parametric test used for comparison is the F test for variances based upon the statistic S_y^2/S_x^2. For the comparisons under normality one might suppose that a better test could be found which takes advantage of the equality of the means. Equality of means is imposed for scale alternatives so that $F = G$ as required for the Kolmogorov - Smirnov null distribution when $\sigma = \tau$. However, even if the means were known there would be a gain of at most one degree of freedom for the optimal statistic in the numerator and denominator and the asymptotic results would be the same.

For the F statistic, if we denote the critical value by d_n we must have $d_n \to \theta^2$ as $n \to \infty$ in order that $\beta_n \to \beta$ with $0 < \beta < 1$. We have

$$\beta_n = P[\, S_y^2/S_x^2 \le d_n \,] = P\left[\frac{S_y^2/\tau^2}{S_x^2/\sigma^2} \le d_n/\theta^2 \right]$$

$$= P[\, F_{n-1,\,n-1} \le d_n/\theta^2 \,]$$

$$= P\left[\frac{F_{n-1,n-1} - \frac{n-1}{n-3}}{\sqrt{\frac{4(n-2)}{(n-3)(n-5)}}} \le \frac{d_n/\theta^2 - (n-1)/(n-3)}{\sqrt{\frac{4(n-2)}{(n-3)(n-5)}}} \right].$$

Using the normal approximation, $\beta_n \to \beta$ provided

$$\frac{d_n/\theta^2 - (n-1)/(n-3)}{\sqrt{\frac{4(n-2)}{(n-3)(n-5)}}} \to z = \Phi^{-1}(\beta),$$

so that $d_n \to \theta^2$. We now show that for fixed θ and the above condition we have

$$\lim -\frac{1}{n} \log \alpha_n = \log \{(1+\theta^2)/2\theta\}. \tag{3.4}$$

Transforming the F distribution to the incomplete beta, (See for example [1, p. 946]) we have under the null hypothesis

$$\alpha_n = P[\, S_y^2/S_x^2 > d_n \,] = \frac{\Gamma(n-1)}{\Gamma^2(\frac{n-1}{2})} \int_0^{x_n} [u(1-u)]^{\frac{n-1}{2}-1}\, du$$

where

$$x_n = \frac{(n-1)}{(n-1)+(n-1)d_n} = \frac{1}{1+d_n} \to \frac{1}{1+\theta^2} < 1/2 \text{ for } \theta > 1.$$

Since $u(1-u)$ is an increasing function on the interval $(0, 1/2)$ we have

$$\alpha_n \leq \frac{\Gamma(n-1)}{\Gamma^2(\frac{n-1}{2})} [x_n(1-x_n)]^{\frac{n-1}{2} - 1} \int_0^{x_n} du \tag{3.5}$$

Also $0 < 1-2u < 1$ for $0 < u < 1/2$ so that

$$\alpha_n \geq \frac{\Gamma(n-1)}{\Gamma^2(\frac{n-1}{2})} \int_0^{x_n} [u(1-u)]^{\frac{n-1}{2} - 1} [1-2u]\, du$$

$$= \frac{\Gamma(n-1)}{\Gamma^2(\frac{n-1}{2})} [x_n(1-x_n)]^{\frac{n-1}{2}} \frac{2}{n-1} . \tag{3.6}$$

Using Stirling's approximation to the gamma functions in (3.5) and (3.6) and $x_n \to 1/(1+\theta^2)$ we derive (3.4).

IV. RELATIVE EFFICIENCIES

According to Section II (2.1), for a fixed alternative and critical values adjusted so that the type II error $\beta_n \to \beta$ $(0 < \beta < 1)$ we have the type I error for the Kolmogorov - Smirnov test going to zero at an exponential rate with increasing sample size

$$\alpha_n = e^{-ne_k[1 + o(1)]}$$

where e_k is the number given on the r. h. s. of (2.1). Similarly for the parametric tests based upon samples of size n^* we have

$$\alpha_{n^*} = e^{-n^*e^*(1+o(1))}$$

where the exponents e^* are given by (3.2) and (3.4) for the t and F tests. Adjusting sample sizes so as to equate errors we have

$$n^*e^*(1+o(1)) = ne_k(1+o(1)).$$

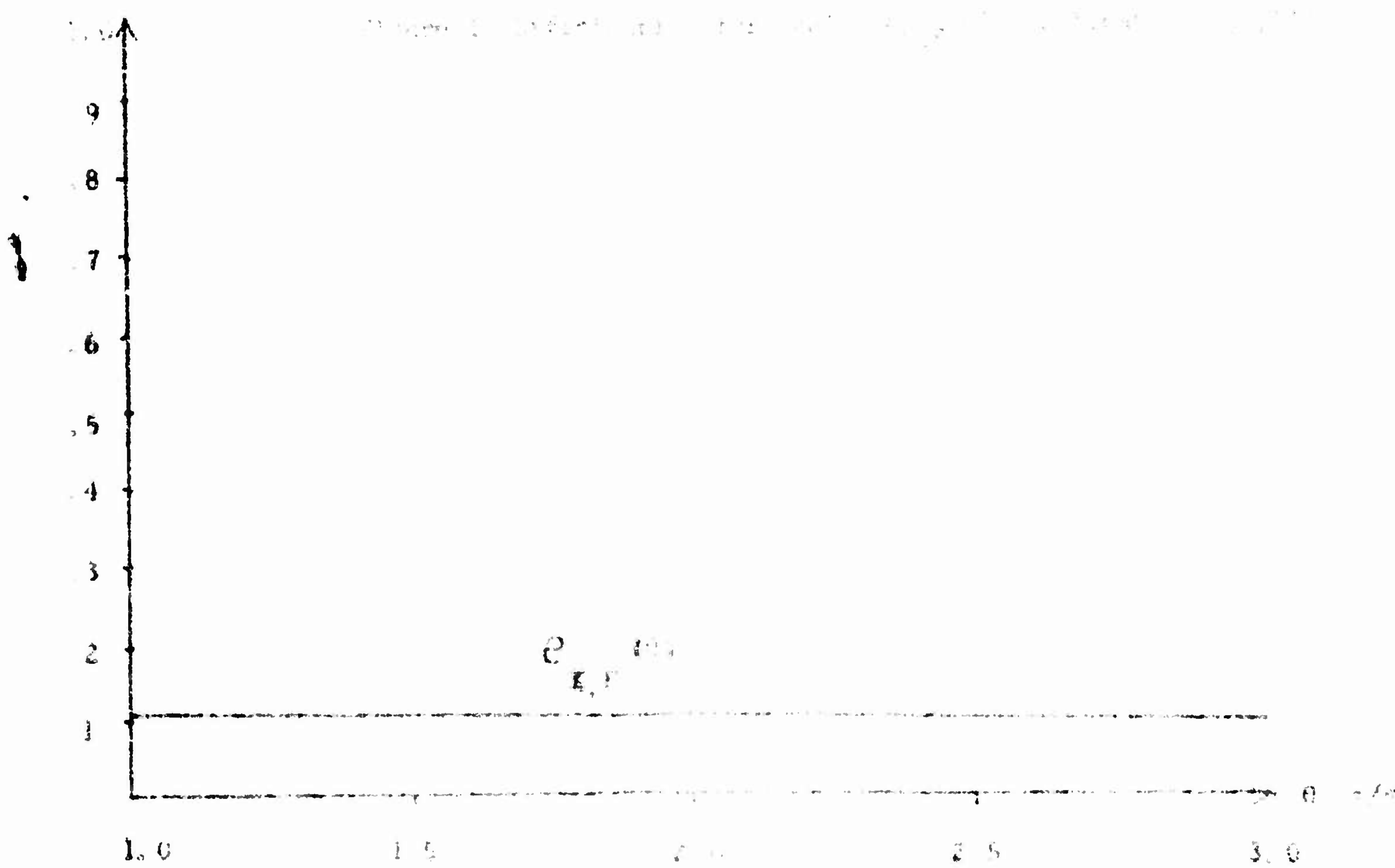
Thus $\lim_{n\to\infty} n^*/n = e_k/e^*$ is a limiting efficiency.

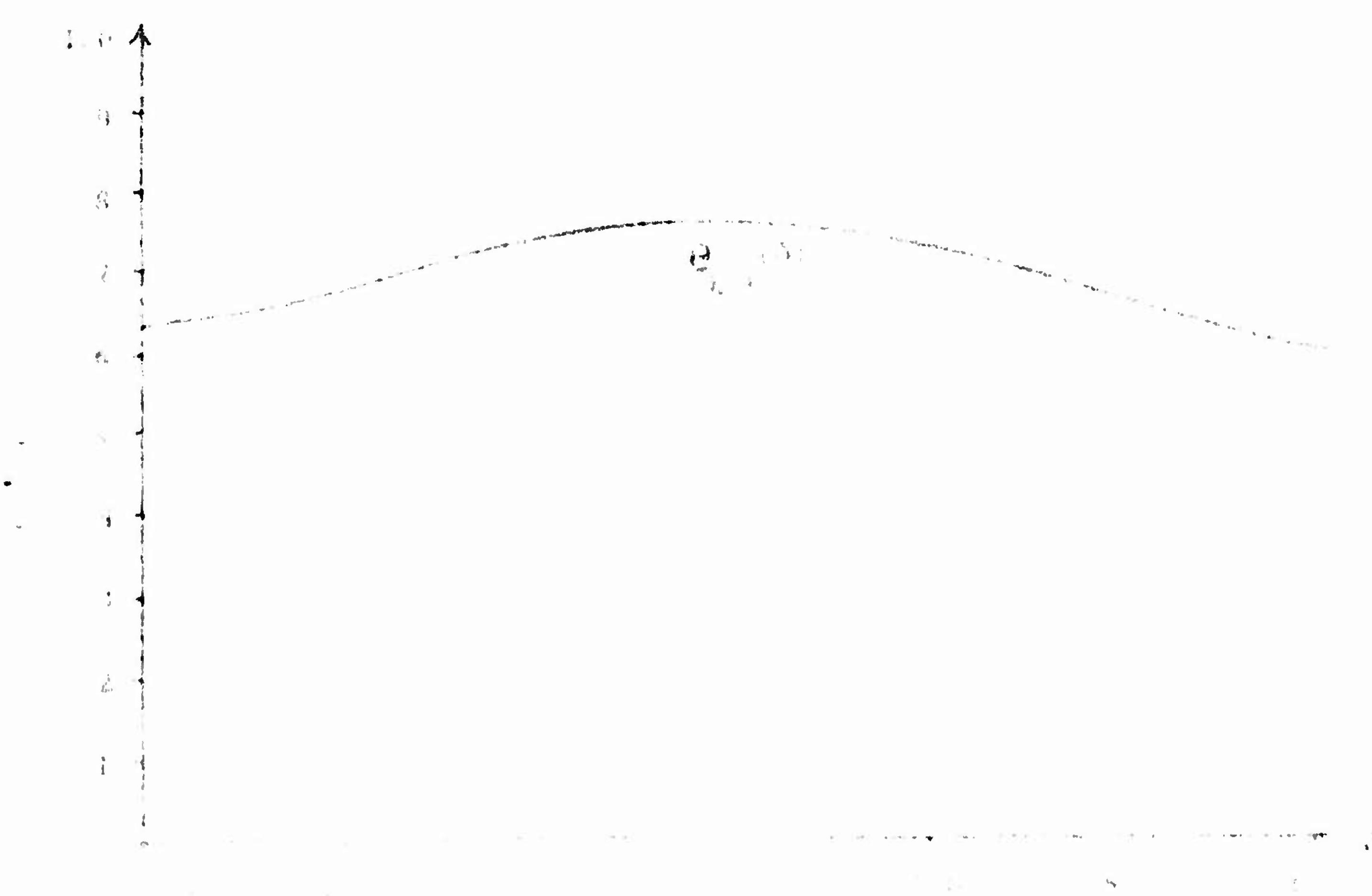
For normal location alternatives the efficiency relative to the t-test is given by the ratio of (2.2) and (3.2)

$$e_{k,t}(\Delta) = \frac{2\Phi(\Delta/2)\ln 2\Phi(\Delta/2) + 2\Phi(-\Delta/2)\ln 2\Phi(-\Delta/2)}{\ln(1+(\Delta/2)^2)} . \tag{4.1}$$

The expression 4.1 is the same as that obtained for the two sample Mood and Brown median test relative to the two sample t for equal samples and is also the same as that given by Bahadur [3, p. 88] for the sign test relative to the one sample t (with θ replaced by $\Delta/2$). The limit of $e_{k,t}(\Delta)$ as $\Delta \to 0$ is $2/\pi$ which is the lower bound derived by Capon for the Pitman efficiency. It is thus conjectured that the Pitman limit is $2/\pi \doteq 637$.

For normal scale alternatives the efficiency relative to the F test is given by


9
8
7
6
5
4
3
2
1
1.0
3.0



$$e_{k,F}(\theta) = \frac{(1-\rho)\ln(1-\rho) + (1+\rho)\ln(1+\rho)}{\ln\{(1+\theta^2)/2\theta\}} \tag{4.2}$$

where $\theta = \tau/\sigma$ and

$$\rho = \Phi\left(\theta\sqrt{\frac{2\ln\theta}{\theta^2-1}}\right) - \Phi\left(\sqrt{\frac{2\ln\theta}{\theta^2-1}}\right).$$

The limit of $e_{k,F}(\theta)$ as $\theta \to 1$ is $(\pi e)^{-1} \doteq .117$ which is the same number obtained by Capon as a lower bound for the Pitman efficiency and by Bahadur [4] using an approximate definition of efficiency for the one sample Kolmogorov test. It is similarly conjectured that this is also the Pitman efficiency value. Tables I and II give values for (4.1) and (4.2).

Because of no convenient closed form expression for the Kolmogorov - Smirnov null distribution with unequal samples, the simple methods of this paper do not appear to extend to cover this case and more complicated methods such as studied by Hoadley [6] and Abrahamson [2] must be used. If the one sided tests are replaced by the two sided tests the expressions (4.1) and (4.2) remain the same.

The small sample interpolated efficiency values of Milton [10, p. II-32] for location seem to indicate that the limiting efficiency is approached by a decreasing sequence. The efficiencies given there for equal samples of size 7 are in the neighborhood of 75%.

ACKNOWLEDGEMENT

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Table I

EFFICIENCY FOR NORMAL SHIFT ALTERNATIVES

Δ	0	.25	.50	.75	1.00	1.25	1.50	1.75	2.00
$e_{k,t}(\Delta)$	.6366	.6393	.6471	.6591	.6742	.6910	.7080	.7240	.7378
Δ	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	
$e_{k,t}(\Delta)$	.7488	.7563	.7600	.7599	.7562	.7491	.7390	.7265	
Δ	4.25	4.50	4.75	5.00	5.25	5.50	5.75	6.00	∞
$e_{k,t}(\Delta)$	.7120	.6960	.6791	.6617	.6441	.6267	.6096	.5931	0

Table II

EFFICIENCY FOR NORMAL SCALE ALTERNATIVES

θ	1.00	1.125	1.250	1.375	1.500	1.625
$e_{k,F}(\theta)$	.1171	.1171	.1172	.1172	.1172	.1172
θ	1.750	1.825	2.000	2.125	2.250	2.375
$e_{k,F}(\theta)$	.1172	.1172	.1172	.1171	.1170	.1169
θ	2.500	2.625	2.750	2.825	3.00	∞
$e_{k,F}(\theta)$	.1167	.1165	.1164	.1162	.1160	0